Applications of Linear Algebra in Combinatorics.

While the probabilistic method is usually useful to construct examples (by randomness) and obtain lower bounds, a common application of Linear Algebra is to prove an upper bound, where we show that objects satisfying certain properties can not be too large. To do so, usually we place the objects by vectors in a linear space of a certain dimension, and we show, the vectors are linearly independent. Thus, # of objects is $\leq$ # dimension. 向量线性无关 ⇒ # ≤ 维数

Odd / Even - Town:

A town has n residents, some clubs. (两个 club 不能含有相同的 members).

- Every club has an odd number of members.
- Every 2 clubs must share an even number of members.

Q: How many clubs can they have?

(1). clubs $A_i = \{i\} \Rightarrow n$ clubs

(2). $n =$ even, $A_i = [n] \setminus \{i\} \Rightarrow n$ clubs

(3). $n =$ even. $A_1 = [n] \setminus \{1\}$, $A_2 = [n] \setminus \{2\}$. $A_i = \{1, 2, i\}$ for $3 \leq i \leq n$

Thm 1: Let $\mathcal{F} \subset 2^{[n]}$ be such that $\begin{cases} \cdot\ |A| \text{ is odd for } \forall A \in \mathcal{F} \\ \cdot\ |A \cap B| \text{ is even for } \forall A, B \in \mathcal{F}. \end{cases}$

Then $|\mathcal{F}| \leq n$.

Proof: For $\forall A \in \mathcal{F}$, define $\vec{I}_A$ be the vector in $\mathbb{Z}_2^n = \{0,1\}^n$, where $\mathbb{Z}_2 = \{0,1\}$ is a finite-field with operation mod 2:

$$\vec{I}_A(i) = \begin{cases} 1, & i \in A \\ 0, & \text{o.w.} \end{cases}$$

$$\Rightarrow (*) \begin{cases} \vec{I}_A \cdot \vec{I}_A = 1. \\ \vec{I}_A \cdot \vec{I}_B = \text{\# of common elements (mod 2)} = 0 \end{cases} \quad \text{in } \mathbb{Z}_2 = \{0,1\}.$$

Let $|\mathcal{F}| = m$. so we have m vectors $\vec{I}_A$ satisfying $(*)$.

Next, we show that they are linearly independent.

Suppose that $\sum_{A\in\mathcal{F}_1} \alpha_A \vec{I}_A = \vec{0}$

For any $B\in\mathcal{F}_1$, $\vec{0} = \vec{0}\cdot\vec{I}_B = \sum \alpha_A(\vec{I}_A\cdot\vec{I}_B) = \alpha_B$ by (*).

$\Rightarrow |\mathcal{F}_1| \le$ # dimension in $\mathbb{Z}_2^n = n$. #

Even/odd - Town:

Club members → Even. Share - odd.

Then, $|\mathcal{F}_1| \le n+1$. Reduction: n, m clubs satisfy Even/odd $\Rightarrow n+1, m$ clubs satisfy Odd/Even Thm1 $\Rightarrow m\le n+1$.

Thm2: $|\mathcal{F}| \le n$.

Proof: By reduction, we have $|\mathcal{F}_1| \le n+1$, Suppose $|\mathcal{F}| = n+1$.

$I_A \in \mathbb{Z}_2^n$ These vector must be linearly dependent.

$\Rightarrow \exists$ non-zero α_A's s.t. $\sum_{A\in\mathcal{F}_1} \alpha_A \vec{I}_A = \vec{0}$.

For $\forall B\in\mathcal{F}_1$, $0 = \vec{0}\cdot\vec{I}_B = \sum_{A\in\mathcal{F}_1}\alpha_A(\vec{I}_A\cdot\vec{I}_B)$ $\begin{cases}\vec{I}_A\cdot\vec{I}_B = 1\\ \vec{I}_B\cdot\vec{I}_B = 0.\end{cases}$

$= \sum_{All\, A\ne B} \alpha_A$

so all α_A's are equal (mod 2)

Since α_A's can't be all zeros

$\Rightarrow \alpha_A = 1$ for $\forall A\in\mathcal{F}_1$.

$\Rightarrow \sum_{A\in\mathcal{F}} \vec{I}_A = \vec{0}$ ... ①

$\Rightarrow n = \sum_{All\, A\ne B} \alpha_A = 0 \Rightarrow n$ is even.

Consider $\{A^c : \forall A\in\mathcal{F}_1\}$.

Clearly, • $|A^c| = n - |A|$ is even

• $|A^c\cap B^c| = n - |A\cup B| = n - (|A| + |B| - |A\cap B|)$ is odd.

Repeat the above proof, we have $\sum_{A^c} \vec{I}_{A^c} = 0$ ... ②

① & ② $\Rightarrow$ $\begin{cases}\sum_{A\in\mathcal{F}_1}\vec{I}_A = \vec{0} & ①\\ \sum_{A\in\mathcal{F}_1}\vec{I}_{A^c} = \vec{0} & ②\end{cases}$

① + ② $\Rightarrow$ $\vec{0} = \sum_{A\in\mathcal{F}_1}(\vec{I}_A + \vec{I}_{A^c}) = |\mathcal{F}_1|\cdot\vec{I} = (n+1)\vec{I} = \vec{I}$ as n is even.

a contradiction! Then $|\mathcal{F}_1| \le n$. #

Even / Even - Town: Thm 3: $|\mathcal{F}| \le 2^{n/2}$

(Hint: For any $A = (**)_{m\times n}$, $\dim(\text{span}\,\vec{A}) + \ker(\vec{A}) = n$.)

Odd / Odd - Town: ?

Fisher's Inequality: Let $\mathcal{F} \subseteq 2^{[n]}$ be such that for some fixed k, $|A\cap B| = k$ holds for any $A, B \in \mathcal{F}$ $(A \neq B)$. Then $|\mathcal{F}| \le n$.

Proof: Let $\vec{I_A}$ be as usual. (over $\mathbb{R}^n$)

$\Rightarrow \vec{I_A} \cdot \vec{I_B} = k \quad \forall A, B \in \mathcal{F}$.

We want to show $\vec{I_A}$ for $\forall A \in \mathcal{F}$ are linearly independent.

Let $\sum_{A\in\mathcal{F}} \alpha_A \vec{I_A} = \vec{0}$, where $\alpha_A \in \mathbb{R}$.

$$\begin{aligned} 0 &= \Big(\sum_{A\in\mathcal{F}} \alpha_A \vec{I_A}\Big)\Big(\sum_{A\in\mathcal{F}} \alpha_A \vec{I_A}\Big) \\ &= \sum_A \alpha_A^2\, \vec{I_A}\cdot\vec{I_A} + \sum_{A\neq B} \alpha_A \alpha_B (\vec{I_A}\cdot\vec{I_B}) \\ &= \sum_A \alpha_A^2 |A| + k \sum_{A\neq B} \alpha_A\alpha_B \\ &= k\Big(\sum_A \alpha_A\Big)^2 + \sum_A \alpha_A^2(|A| - k) \geq 0. \quad \cdots (*) \end{aligned}$$

Since all subsets $A \in \mathcal{F}$ are of size at least k and at most one subset is of size k, $(|A| \geq k)$

$|A| = k$

$\forall A \neq B \in \mathcal{F},\ A \subsetneq B$.

from the inequality $(*)$, we see that

$$\forall A \in \mathcal{F},\quad \alpha_A^2(|A| - k) = 0, \qquad \sum_A \alpha_A = 0.$$

$\Rightarrow \forall A \in \mathcal{F},\ \alpha_A = 0$ or $|A| = k$, $\sum \alpha_A = 0$.

Let A^* be the subset of size k (if exist). Then $\forall A \in \mathcal{F} - \{A^*\}$, $\alpha_A = 0$.

so that $\sum_A \alpha_A = \alpha_{A^*} = 0$. Then $\forall A \in \mathcal{F}$, $\alpha_A = 0$.

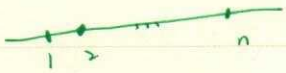
$\Rightarrow$ vectors $\{\vec{I_A} : A \in \mathcal{F}\}$ are linearly independent.

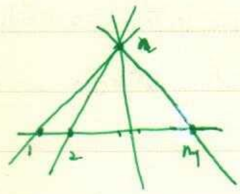
$\Rightarrow |\mathcal{F}| = \#$ vectors $\vec{I_A}$'s $\le \#$ dimension in $\mathbb{R}^n = n$. #

Lemma 1: Suppose P is a set of n points in $\mathbb{R}^2$, Then either they are in a line or they define at least n lines.

$\longrightarrow$

1 or $\geq n$.





Proof: Let L be the collection of lines defined by the pair of P.
For $\forall x_i \in P$, define L_i to be the family of lines from L containing x_i. $\Rightarrow L_i \subseteq L$, $|L_i \cap L_j| = 1$. $L_i \wedge L_j =$ line $x_i x_j$.
By Fisher's Inequality, $|P| = \#$ of L_i's $\leq |L|$.
$\Rightarrow |L| \geq n$.
O.W., $L_1 = L_2 = \cdots \Rightarrow |L| = 1$. #

小结

• Odd/Even	Even/Odd
Let $\mathcal{F} \subseteq 2^{[n]}$, be such that	Let $\mathcal{F} \subseteq 2^{[n]}$, be such that
• $\|A\|$ is odd. $\forall A \in \mathcal{F}$	• $\|A\|$ is even. $\forall A \in \mathcal{F}$
• $\|A \cap B\|$ is even, $\forall A \neq B \in \mathcal{F}$	• $\|A \cap B\|$ is odd, $\forall A \neq B \in \mathcal{F}$.
the $\|\mathcal{F}\| \leq n$	then $\|\mathcal{F}\| \leq n$.

• Fisher's Inequality:
$\mathcal{F} \subseteq 2^{[n]}$, $\forall$ distinct $A, B \in \mathcal{F}$, $|A \cap B| = k$, then $|\mathcal{F}| \leq n$.
$\Rightarrow$ Lemma 1: n points in $\mathbb{R}^2$, $\Rightarrow$ 1 lines or $\geq n$ lines.

Lemma 2: For any k, let G be a graph whose vertices are triples in $\binom{[k]}{3}$ and $A \underset{G}{\sim} B$ is an edge of $|A \cap B| = 1$. Then G does not have any clique or independent set of size more than k.

Rmk: $\Rightarrow R(k+1, k+1) > \binom{k}{3}$ which is much worse than the random construction $(R(k,k) > c \cdot k \cdot 2^{\frac{k}{2}})$.

Proof: • Consider a maximum clique, whose vertices say $A_1, A_2, \cdots A_m$.

so $|A_i \cap A_j| = 1 \quad \forall\ 1 \le i < j \le m$.

By Fisher's Ineq. max clique size $= m \le k$.

• Consider a maximum independent set, say $B_1, B_2, \cdots B_t$.

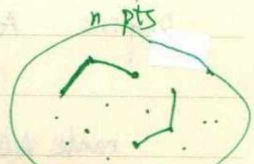
so $|B_i \cap B_j| = 0$ or $2 \quad \forall\ 1 \le i < j \le t$, where each $B_i \subseteq [k]$ of odd size,

By Odd/Even-Town, max independent set size $= \alpha(G) = t \le k$. #

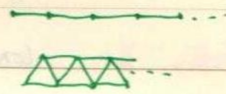
1-distance Problems

• Problem 1: Given n points in $\mathbb{R}^2$,

How many pairs of points have distance 1?



Def Given a graph H, $ex(n, H)$ = maximum number of edges in a n-vertex H-free graph.



$\Rightarrow$ Mental's Thm: $ex(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

$ex(n, K_{2,2}) = O(n^{\frac{3}{2}})$, $ex(n, K_{3,3}) = O(n^{\frac{5}{3}})$

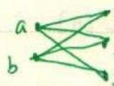
Exercise: $ex(n, K_{2,3}) = O(n^{\frac{3}{2}})$.

Thm 1: There are at most $O(n^{\frac{3}{2}})$ pairs at distance 1.

Proof: Define a graph G as follows:

$V(G) = \{n \text{ points}\}$ & $a \underset{G}{\sim} b$ iff $d(a, b) = 1$.

Claim: Such G is $K_{2,3}$-free.



Why? The neighbors of point a must lie on the circle with center a and radius 1.

三点定圆.
两个圆至多2个交点.

But any such 2 circles can only intersect at most 2 points. This tells us any 2 vertices at most have 2 common neighbors. so it is $K_{2,3}$-free.

By Exercise, $ex(n, K_{2,3}) = O(n^{\frac{3}{2}})$. #

• Problem 2: How many points in $\mathbb{R}^n$. s.t. the distance between any 2 points is 1.

Thm 2: In $\mathbb{R}^n$, there are at most $n+1$ such points.

Proof: We have, say m points in $\mathbb{R}^n$.

we many assume one of them is $\vec{0}$, and then the left $m-1$ points can be viewed as vectors $\vec{v}_1 \cdots \vec{v}_{m-1} \in \mathbb{R}^n$

$\Rightarrow \quad \vec{v}_i \cdot \vec{v}_i = 1 \quad \& \quad \vec{v}_i \cdot \vec{v}_j = \frac{1}{2} \quad \forall i \neq j$.

$\Uparrow$

$1 = \|\vec{v}_i - \vec{v}_j\| = \|v_i\|^2 + \|v_j\|^2 - 2\vec{v}_i \cdot \vec{v}_j$

Define $A = \begin{pmatrix} \vec{v}_1 \\ \vdots \\ \vec{v}_{m-1} \end{pmatrix}_{(m-1)\times n}$. $AA^T = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & \frac{1}{2} \\ \frac{1}{2} & 1 & \ddots & \frac{1}{2} \\ \vdots & & \ddots & \vdots \\ \frac{1}{2} & \frac{1}{2} & \cdots & 1 \end{pmatrix}_{(m-1)(m-1)}$ AA^T 满秩.

$\operatorname{rank} AA^T = m-1 \leq \operatorname{rank} A \leq n \quad \Rightarrow m \leq n+1$. #

Erdös' Thm: Construct n points in $\mathbb{R}^2$, with $n^{1+\frac{1}{\log(\log n)}}$ pairs at distance 1.

Open Problem: Find an example of n points in $\mathbb{R}^2$ with n^{1+c} pairs at distance 1 for some constant $c > 0$.

Spaces of polynomials:

Insted of consider vectors, we also can define certain polynomials, as polynomials of certain degree also form a vector space.

$A \subset [n] \to \vec{I}_A \to f_A(x) = \sum_{i \in A} x_i \in \operatorname{span}\{x_1, \cdots x_n\}$ = a vector space of n-dimension.

Lemma: Let $f_i : \Omega \to \mathbb{F}$ be functions for $i = 1, 2, \cdots m$.

若 $\exists \ v_i \in \Omega$, s.t. $\begin{cases} (a). \ f_i(v_i) \neq 0, \ \forall i \\ (b). \ f_i(v_j) = 0 \quad \text{for } 1 \leq j < i \leq m. \end{cases}$

then $f_1 \cdots f_m$ are linearly independent over $\mathbb{F}^{\Omega}$. #

2-distance set:

Def: 2-distance set is a set of points in $\mathbb{R}^n$ whose pairwise distance is either c or d, for some $c, d > 0$.

Exercise: construct such a set of size $\binom{n}{2}$.
(consider all 0/1 vectors with exactly 2 1's.

Exercise: construct such a set of size $\binom{n+1}{2}$.

Thm: Any 2-distance-set in $\mathbb{R}^n$ has at most $\frac{1}{2}(n+1)(n+4)$ points.

Proof: Let $A \subseteq \mathbb{R}^n$ be such a set with distance $c > 0$ and $d > 0$.

For each $\vec{a} \in A$, define

$$f_{\vec{a}}(\vec{x}) = (\|\vec{x}-\vec{a}\|^2 - c^2)(\|\vec{x}-\vec{a}\|^2 - d^2) \quad \text{over } \vec{x} \in \mathbb{R}^n.$$

$\Rightarrow$ For $\forall \vec{b} \in A - \{\vec{a}\}$, $f_{\vec{a}}(\vec{b}) = (\|\vec{b}-a\|^2 - c^2)(\|\vec{b}-a\|^2 - d^2) = 0$.

$\|\vec{b}-a\| = c$ or d.

and $f_{\vec{a}}(\vec{a}) = c^2 d^2 > 0$.

By Lemma, $\{f_{\vec{a}} : \vec{a} \in A\}$ are linearly independent. ✓

We want to bound the dimension of the vector space (containing all polynomials $f_{\vec{a}}$).

$$f_{\vec{a}}(\vec{x}) = (\sum (x_i - a_i)^2 - c^2)(\sum (x_i - a_i)^2 - d^2)$$

$$= (\sum x_i^2 - \sum 2a_i x_i + \sum a_i^2 - c^2)(\sum x_i^2 - \sum 2a_i x_i + \sum a_i^2 - d^2)$$

this can be expressed as linear combination of the follows:

$$V = \{ \underbrace{(\sum x_i^2)^2}_{1},\ \underbrace{x_j(\sum x_i^2)}_{n},\ \underbrace{x_i x_j}_{n+\binom{n}{2}},\ \underbrace{x_j}_{n},\ \underbrace{1}_{1} \ \text{ where } 1 \le i, j \le n \}.$$

$$\dim V = 1 + n + n + \binom{n}{2} + n + 1 = 2(n+1) + \frac{n(n+1)}{2} = \frac{1}{2}(n+1)(n+4).$$ ✓

The polynomials $f_{\vec{a}}$ reside in a vector space of dimension $\frac{1}{2}(n+1)(n+4)$.

So, $|A| \le \frac{1}{2}(n+1)(n+4)$. #